Conditional expectations and martingales

Def. Let $\mathcal{A} \subset \mathcal{B}$ - σ-algebras in Ω.

f - $\mathcal{B}$-measurable, $E(|f|) < \infty$. ($f \in L^1$)

Then $g = E(f|\mathcal{A})$ is the unique $\mathcal{A}$-measurable function

conditional expectation

such that $\int_A g\, dP = \int_A f\, dP \quad \forall A \in \mathcal{A}$.

Exists because $A \to \int_A f\, dP$ is an absolutely continuous measure on $\mathcal{A}$, so we can use Radon-Nykodim.

Example For dyadic filtration,

$$E(f|\mathcal{B}_n) = \sum_{k=0}^{2^n-1} 2^n \left(\int_{k2^{-n}}^{(k+1)2^{-n}} f\, dx \right) \mathbb{1}_{[k2^{-n}, (k+1)2^{-n})}.$$

Some easy properties:

0) Linearity. $E(\alpha f + \beta g|\mathcal{A}) = \alpha E(f|\mathcal{A}) + \beta E(g|\mathcal{A})$

1) $\mathcal{A} \subset \mathcal{B} \Rightarrow E(E(f|\mathcal{B})|\mathcal{A}) = E(f|\mathcal{A})$

(compare $\int_A$ of two sides).

2) If h is $\mathcal{A}$ measurable, then

$E(hf|\mathcal{A}) = h E(f|\mathcal{A})$. In particular, $E(h|\mathcal{A}) = h$.

Proof. True for $h = \mathbb{1}_{A_0}$, $A_0 \in \mathcal{A}$, since $\forall A \in \mathcal{A}$, $A_0 \cap A \in \mathcal{A}$.

So true for step functions. Pass to the limit ∎

3) If f is independent of $\mathcal{A}$ then

$E(f|\mathcal{A}) = E(f)$.

(independent $\Rightarrow \forall A \in \mathcal{A}$ $\int_A f = E(f)P(A)$)

Def Let $(X_t)_{t\in I}$ be a real-valued $(\mathcal{F}_t)$-adapted process.
$\forall t \in I$ $E(|X_t|) < \infty$.

X_t is called <u>martingale</u> / <u>sub martingale</u> / <u>super martingale</u> if $\forall s \le t \in I$:

$s < t$

- martingale: $E(X_t | \mathcal{F}_s) = X_s$
- sub martingale: $E(X_t | \mathcal{F}_s) \ge X_s$
- super martingale: $E(X_t | \mathcal{F}_s) \le X_s$.

<u>martingale</u> = sub + super.

Examples. 1) f - $\mathcal{F}_\infty$ -measurable, $E(|f|) < \infty$ (L^1).

Let $f_s := E(f | \mathcal{F}_s)$ - martingale.

$(E(f_t | \mathcal{F}_s) = E(E(f | \mathcal{F}_t) | \mathcal{F}_s) = E(f | \mathcal{F}_s) = f_s)$.

For dyadic:

$$f_n(x) = 2^n \int_{k2^{-n}}^{(k+1)2^{-n}} f(t)\,dt \quad \text{if} \quad k2^{-n} \le x < (k+1)2^{-n}.$$

2) <u>Brownian Motion</u>.

$$E(B_t | \mathcal{F}_s) = E(\underbrace{B_t - B_s}_{\text{independent of } \mathcal{F}_s} + B_s | \mathcal{F}_s) = E(B_s | \mathcal{F}_s) = B_s.$$

3) $B_t^2 - t$.

$$E(B_t^2 - t | \mathcal{F}_s) = E(\overbrace{(B_t - B_s)^2}^{\text{indep.}} + 2B_t B_s - t - B_s^2 | \mathcal{F}_s) =$$

$$t - s - t + B_s E(2B_t - B_s | \mathcal{F}_s) = 2B_s^2 - B_s^2 - s = B_s^2 - s.$$

4) $Y_{t,\alpha} = e^{\left(\alpha B_t - \frac{\alpha^2 t}{2}\right)}$

Markov

$$E(Y_{t,\alpha} \mid \mathcal{F}_s) = E\left(e^{\alpha(B_t - B_s) - \frac{\alpha^2}{2}(t-s)} \cdot e^{\alpha B_s - \frac{\alpha^2}{2}s} \mid \mathcal{F}_s\right) =$$

$$e^{\alpha B_s - \alpha^2/2\, s}\, E\left(e^{\alpha B_{t-s} - \frac{\alpha^2}{2}(t-s)}\right) = Y_{s,\alpha}.$$

5) Super-example – first instance of Itô integral

Discrete time.

H_n–bounded predictable process (i.e. H_n is $\mathcal{F}_{n-1}$ measurable)

X_n – martingale.

Let

$$Y_0 = X_0, \qquad Y_n = Y_{n-1} + H_n(X_n - X_{n-1}).$$

Then Y_n is also martingale.

Indeed: $E(Y_n \mid \mathcal{F}_{n-1}) = E(Y_{n-1} + H_n(X_n - X_{n-1}) \mid \mathcal{F}_{n-1}) =$

$$Y_{n-1} + H_n E((X_n - X_{n-1}) \mid \mathcal{F}_{n-1}) = Y_{n-1}$$

Notation: $Y_n = (H \cdot X)_n$.

Corollary T – stopping time $X_n^T := X_{\min(n,T)}$ is a martingale

Proof. Let $H_n := \mathbb{1}_{T \geq n} = 1 - \mathbb{1}_{T \leq n-1}$ – predictable.

$$X_n^T = (H \cdot X)_n \quad \text{since} \quad X_n^T - X_{n-1}^T = \mathbb{1}_{T \geq n}(X_n - X_{n-1})$$

Thm (Discrete Optional Stopping Time Thm)

Let X_n be a discrete-time martingale,

$S \leq T$ – two bounded stopping times

Let $\mathcal{F}_S := \{A \in \mathcal{F}_\infty : \forall n \quad A \cap \{S = n\} \in \mathcal{F}_n\}$.

Then $X_S = E(X_t \mid \mathcal{F}_S)$.

Proof. Let $M \geq T \geq S$, $M \in \mathbb{N}$. WLOG $X_0 \equiv 0$ - (subtract)

Let $H_n := \mathbb{1}_{T \geq n} - \mathbb{1}_{S \geq n}$.

Take $n > M$, then $(H \cdot X)_n = X_T - X_S$.

so $E(X_T) - E(X_S) = E((H \cdot X)_n) = E(X_0) = 0$.

Fix $B \in \mathcal{F}_S$. Consider
$$\left.\begin{aligned} \tilde{S} &:= S \mathbb{1}_B + M \mathbb{1}_{B^c} \\ \tilde{T} &:= T \mathbb{1}_B + M \mathbb{1}_{B^c} \end{aligned}\right\} \text{stopping times by definition of } \mathcal{F}_S.$$

$\tilde{S} \leq \tilde{T} \leq M$ - stopping times.

so $E(X_T \mathbb{1}_B) = E(X_{\tilde{T}}) - M\,P(B^c) = E(X_{\tilde{S}}) - M\,P(B^c) =$
$E(X_S \mathbb{1}_B)$. ■

Remark. Same way! if (X_t) - submartingale, (super-)

$$E(X_T \mid \mathcal{F}_S) \geq X_S.$$
$$(\leq)$$